7.1 Path Integrals

<u>Motivation</u>

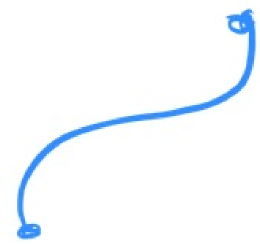
$$f: \mathbb{R}^3 \to \mathbb{R}$$

$$c: [a,b] \to \mathbb{R}^3$$

concrete example:

$f(\vec{x})$ temperature at $\vec{x}$

$c[a,b)$ wire in $\mathbb{R}3$



calculate heat energy in wire

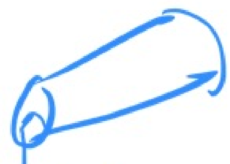
heat energy $\approx$ temperature $\times$ volume $\times$ constant

↑ depends on material

easy case:

- $\overbrace{\text{temperature}}^{f}$ constant
- wire ~ straight line

heat energy $H = f \cdot \text{length wire} \times \overbrace{\underbrace{\text{area of } \circ}_{\text{cross section of wire}}}^{\text{constant}}$



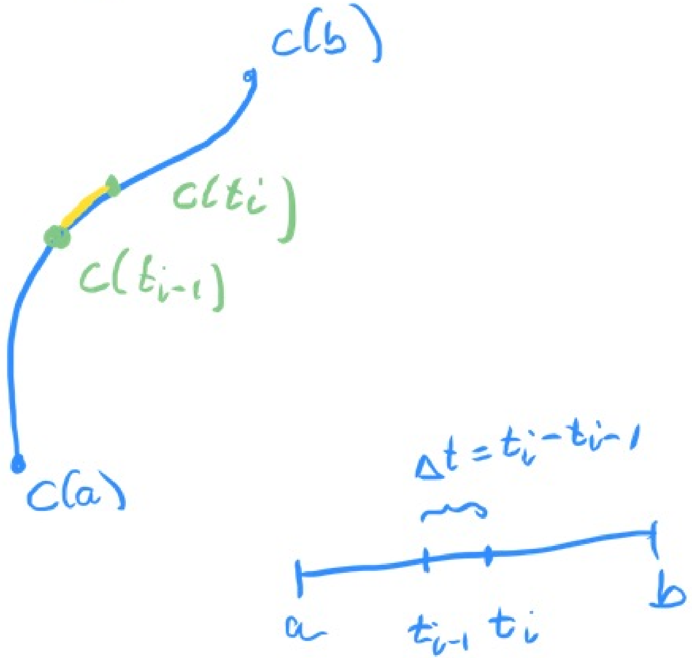
- for simplicity: const. = 1
- also absorb area of cross-section in constant

simplified formula: $\boxed{H = \underset{\uparrow \atop \text{constant}}{f} \cdot \underset{\uparrow \atop \text{straight line}}{\text{length of wire}}}$

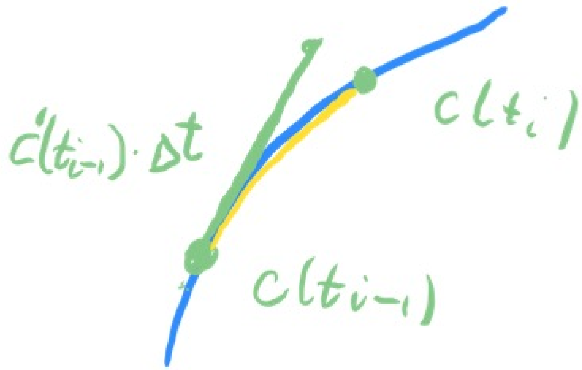
<u>more difficult</u>:

- f not constant
- wire has complicated shap

We reduce the general case to the previous special case as follows



magnify



Crucial Point: we approximate the yellow segment of the curve by the tangent vector $c'(t_{i-1}) \Delta t$

$\Rightarrow$ work from $c(t_{i-1})$ to $c(t_i)$ $\approx$ $f(c(t_{i-1})) \cdot \| c'(t_{i-1}) \| \Delta t$

approximately constant for small Δt

hence: work on curve $\approx \sum_{i=1}^{n} f(c(t_{i-1})) \cdot \|c'(t_{i-1})\| \Delta t$

if we let Δt shrink to 0 (and $n \to \infty$)

the sum $\rightsquigarrow \int_a^b f(c(t)) \|c'(t)\| \, dt$

<u>Def</u> Let $f: \mathbb{R}^n \to \mathbb{R}$, $c: [a,b] \to \mathbb{R}^n$ a curve

Then the path integral is defined by

$$\boxed{\int_C f \, ds = \int_a^b f(c(t)) \|c'(t)\| \, dt}$$

Examples:

① Let $f(x,y,z) = (x^2+y^2)z$

Let $c(t) = (\cos t, \sin t, t)$, $0 \le t \le 2\pi$

Calculate $\int_C f\,ds$

Solution: $$= \int_0^{2\pi} f(\underset{x}{\cos t}, \underset{y}{\sin t}, \underset{z}{t})\, \| \underbrace{(-\sin t, \cos t, 1)}_{=c'(t)} \|\, dt$$

$$= \int_0^{2\pi} (\underset{x^2}{\cos^2 t} + \underset{y^2}{\sin^2 t})\, \underset{z}{t}\, \sqrt{(-\sin t)^2 + (\cos t)^2 + 1^2}\, dt$$

$$= \int_0^{2\pi} t \cdot \sqrt{2}\, dt = \frac{\sqrt{2}}{2} t^2 \Big|_0^{2\pi} = \boxed{2\sqrt{2}\,\pi^2}$$

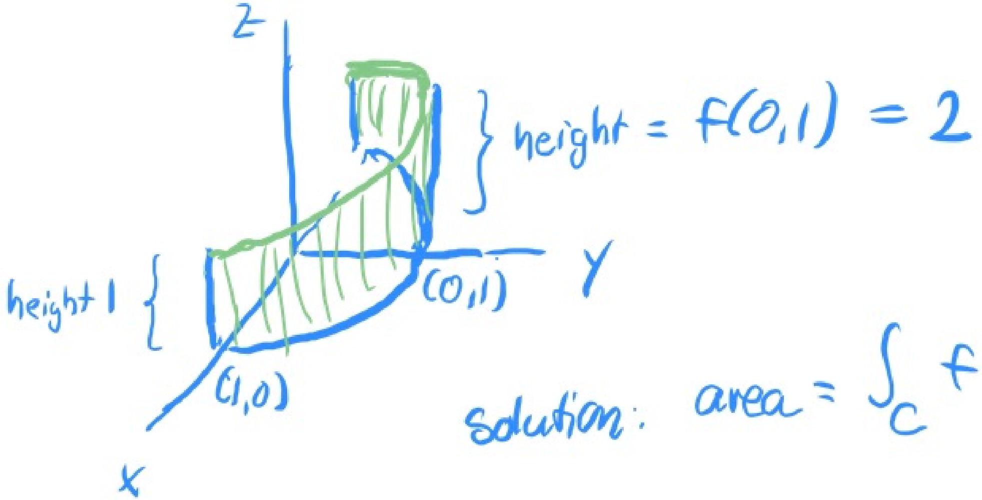
(2) For $n=2$, path integral can be interpreted as area of a fence of height $f(x,y)$

<u>Ex.</u> Calculate area of fence above semicircle

$$c(t) = (\cos t, \sin t), \quad 0 \le t \le \pi$$

whose height at point (x,y) is given by

$$f(x,y) = 1 + y$$



solution: area $= \int_C f\, ds = \int_0^{\pi} f(\cos t, \sin t)\, \|(-\sin t, \cos t)\|\, dt$

$$= \int_0^{\pi} (1 + \sin t) \cdot 1 \, dt$$

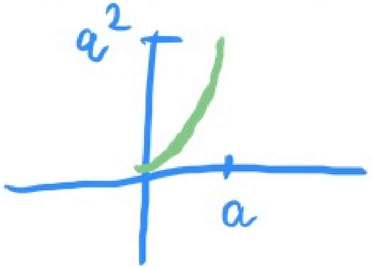
$$= t - \cos t \Big|_0^{\pi} = \boxed{\pi + 2}$$

③ path integral can be used to calculate length of a curve.

$c : [a,b] \to \mathbb{R}^n$

$$\text{length of curve} = \int_c 1 \, ds = \int_a^b \|c'(t)\| \, dt$$

Example: Calculate length of parabola from $(0,0)$ to (b,b^2)



$$\boxed{\begin{aligned} c(t) &= (t, t^2) \\ 0 &\leq t \leq b \end{aligned}}$$

sol. length of curve

$$= \int_0^b \|c'(t)\| \, dt$$

$$= \int_0^b \sqrt{1^2 + 4t^2} \, dt$$

$$= \int_0^b 2 \sqrt{t^2 + (1/2)^2}\, dt =$$

$=$
integral tables book nr. 43
$a = 1/2$

$$= t\sqrt{t^2 + 1/4} + \left(\frac{1}{2}\right)^2 \log | t + \sqrt{t^2 + 1/4} \Big|_0^b$$

$$= b\sqrt{b^2+1/4} + \frac{1}{4} \log | b + \sqrt{b^2 + 1/4} | - \frac{1}{4} \log \sqrt{1/4}$$

$$= \boxed{b\sqrt{b^2+1/4} + \frac{1}{4} \log | b + \sqrt{b^2+1/4} + \frac{1}{4} \log 2}$$